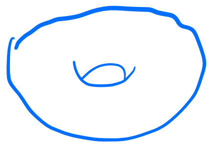
Talk: Maximal Tori on Lie Groups

(Refer to notes from John Morgan)

I. What is a Torus?:

- A Torus is $(S^1)^n = \underbrace{S^1 \times \dots \times S^1}_{n \text{ times}}$ (Cartesian Product of Circles)

  S^1 – Unit Circle.

- Visualization:  Looks like a Donut.

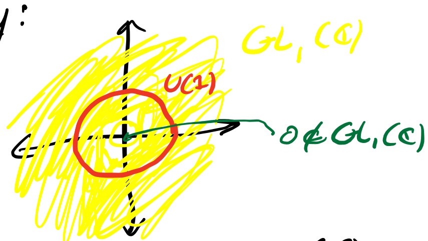
II. Examples of Tori in Lie Groups:

(i) In $GL_1(\mathbb{C}) := \{1 \times 1 \text{ invertible matrices}\}$

$$= \{[a] \mid a \neq 0\}$$

$U(1) = \{a \in \mathbb{C} \mid |a| = 1\}$

Pictorially:



- This is the largest torus in $GL_1(\mathbb{C})$.
  - Why?

    Dimensionality: you need can't have a set of 10x10 matrices for example living inside of a set of 1x1 matrices.

(ii) $GL_n(\mathbb{C})$

Does what we did in $GL_1(\mathbb{C})$ generalize to $GL_n(\mathbb{C})$?

- Fortunately, it does.

$GL_n(\mathbb{C}) := \{n \times n \text{ invertible matrices}\}$

So in $GL_1(\mathbb{C})$ we had $U(1)$

↳ In $GL_n(\mathbb{C})$ what do we have?

- Any thoughts?

• <u>Idea:</u> We can take $(U(1))^n$ in $GL_n(\mathbb{C})$.
$(U(1))^n = \{(a_1, a_2, \ldots, a_n) \mid |a_i| = 1\}$

- Problem: In $GL_1(\mathbb{C})$ case, we had a 1×1 matrix. It is easy to realize $U(1)$ as a matrix. Now, we have an $n\times n$ matrix or in other words n^2 entries to put n entries into.

Where do we put these entries?

<u>Solution:</u> Place the entries across the diagonal.

$$\left\{A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \;\middle|\; |a_i| = 1\right\}$$ det $A \neq 0$
Hence $A \in GL_n(\mathbb{C})$.

Makes sense why this would be a maximal torus.

(iii) $T = \left\{ \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \;\middle|\; \theta_1 + \ldots + \theta_n = 0 \right\} \subseteq SU(n)$

$SU(n)$ – Special Unitary Group

- $U(n)$ + determinant 1.

<u>Notice:</u> In all of these examples, these tori were the largest such that could fit in the particular matrix group.

Begs the question:
- Is there an upper bound?

<u>Def (4.6)</u> A <u>maximal torus</u> $T \subset G$ is a subgroup which is a torus, such that if $T \subset U \subset G$ and U is a torus, then $T = U$.

<u>Note:</u> G is a Compact Connected Lie Group.
- If G is not compact, then you need not have any non-trivial tori.

<u>Prop (4.8):</u> Any subtorus of G is contained in a maximal torus.

Intuitively, this seems obvious. Your tori are all of this same general form. The only real difference is the dimension.

<u>Pf</u> Consider a strictly increasing sequence of subtori $T_1 \subset T_2 \subset \ldots \subset G$.
Then, $L(T_1) \subset L(T_2) \subset \ldots \subset L(G)$ is a strictly increasing sequence, and so is finite. $\square$

<u>The takeaway:</u> These tori are nested in each other.
The maximal tori are loosely "outermost" tori, taking up the most space in G.

We now somewhat get what tori/maximal tori are.
Why are they so important?

Thm (2.19) A connected Abelian Lie Group G has the form $T^a \times R^b$.

Corrollary (2.20) A lie group which is compact, connected, and Abelian is a Torus.

Now we know that any compact connected abelian Lie Group is in fact a Torus.

Def (4.1) Let G be a Topological group and let $g \in G$.
Let H be the subgroup generated by g.
Then g is a generator of G if $cl(H) = G$,
where cl denotes the closure.
G is monogenic (or monothenic) if it has a generator.

• Important Note: Monogenic $\Longrightarrow$ Abelian.

Prop (4.3) The torus T^K is monogenic and generators are dense in T^K.

Proof. Let $U_1, U_2, \ldots$ be a countable base for the open sets of T^k. Let $T^k = R^k/Z^k$ have co-ordinates $(x_1, \ldots, x_k)$.

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Then a cube is a set $\{x \in T^k;\ |x_i - \xi_i| \leq \epsilon\}$ for some fixed point ξ, and real $\epsilon > 0$. Let C_0 be any cube. Then we will define a descending sequence of subcubes whose intersection will be a generator.

Suppose, inductively, that we have defined $C_0 \supset C_1 \supset \ldots \supset C_{m-1}$ and that C_{m-1} has side 2ϵ. Then there is an integer $N(m)$ such that $N \cdot 2\epsilon > 1$, so that the image of C_{m-1} under multiplication by N is T^k. We can find $C_m \subset C_{m-1}$ such that $N \cdot C_m \subset U_m$.

Let $g \in \cap_m C_m$. Then $g^{N(m)} \subset U_m$, so g is a generator of T^k.

What did we gain from this?
- Well, the fact that our group G is or isn't monogenic is somewhat besides the point.
- Our tori will always be monogenic and therefore also abelian.

• We should be able to leverage this fact to determine if G is monogenic.

Prop (4.4)

4.4 PROPOSITION. Let G be an Abelian topological group, with $T^k \subset G$ such that $G/T^k = Z_m$. Then G is monogenic.

Proof. Let t be a generator of T^k. Choose $u \in G$ to project to a generator of Z_m. Then $mu \in T^k$ and $t - mu \in T^k$. T^k is divisible, so there is $s \in T^k$ with $ms = t - mu$. Take $g = u + s$. Then $mg = m(u + s) = t$, so the powers of g are dense in T. Translating by rg, the powers of g are dense in the coset of T containing ru. This gives all cosets.

Now we have a way to confirm if G is abelian.
• Knowing a group is abelian enables us to do so much more with groups or at least simplify certain problems that arise.

Def (4.11)| The integer lattice of $L(T)$ is $\exp^{-1}(e)$ where $\exp : L(T) \to T$.

Prop (4.12)| $L(G) = G_e$ splits as a T-space in the form $V_0 \oplus \sum_{i=1}^{m} V_i$, where T acts on V_0 trivially, $\dim V_i = 2$ for $i > 0$ and T acts on V_i as $\begin{bmatrix} \cos 2\pi \theta_i(t) & -\sin 2\pi \theta_i(t) \\ \sin 2\pi \theta_i(t) & \cos 2\pi \theta_i(t) \end{bmatrix}$

Here, $\theta_i : T \to \mathbb{R}/\mathbb{Z}$ is given by the linear form $\theta_i : L(T) \to \mathbb{R}$ taking integer values on the integer lattice, and no θ_i is zero.

Prop (4.14)|: T is maximal $\Longleftrightarrow V_0 = L(T)$.

Notation (1.7)|: Let G be a Lie Group, with a unit e. Then we write $L(G)$ for G_e and $L(f)$ for $f_*|G_e$. Then L is a functor. ?

What does $L(T)$ mean?

Tangent space of that Group at the identity.
So in this tangent space of the torus at the identity

Pf| It is clear that $L(T) \subset V_0$.

(i) Suppose that $V_0 = L(T)$ and $T \subset T'$. Then, $L(T) \subset L(T') \subset V_0' \subset V_0$, so $L(T) = L(T')$ and $T = T'$.

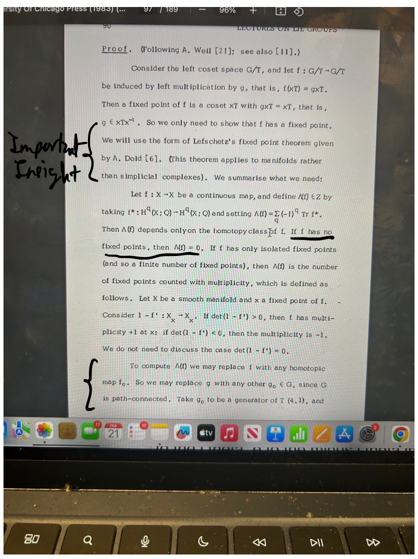
(ii) Suppose $V_0 \neq L(T)$. Then there is $X \in V_0$, $X \notin L(T)$. Now $\exp(tX)$, for $t \in \mathbb{R}$, is a 1-parameter subgroup H of G on which T acts trivially, and which is not contained in T. Therefore, the subgroup generated by T and H is a connected Abelian subgroup strictly containing T, so T is not maximal. $\square$

Corollary (4.15)|: $(\dim G - \dim T)$ is even.

We have found some pretty interesting results regarding Lie Groups. However, you may be curious about what this all has to do with the prior talk on Lefschetz Fixed Point Theorem. Here is an application of the theorem.

Thm (4.21) Let $T \subset G$ be a maximal torus. Then any $g \in G$ is contained in a conjugate of T.

(Proof Takes too long to go over in full. Give the main idea/strategy)



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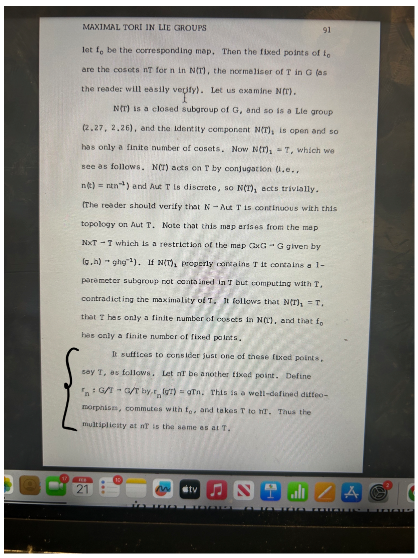
Proof. (Following A. Weil [21]; see also [11].)

Consider the left coset space G/T, and let $f : G/T \to G/T$ be induced by left multiplication by g, that is, $f(xT) = gxT$. Then a fixed point of f is a coset xT with $gxT = xT$, that is, $g \in xTx^{-1}$. So we only need to show that f has a fixed point. We will use the form of Lefschetz's fixed point theorem given by A. Dold [6]. (This theorem applies to manifolds rather than simplicial complexes). We summarise what we need:

Important Insight

Let $f : X \to X$ be a continuous map, and define $\Lambda(f) \in Z$ by taking $f^* : H^q(X; Q) \to H^q(X; Q)$ and setting $\Lambda(f) = \sum_q (-1)^q \operatorname{Tr} f^*$. Then $\Lambda(f)$ depends only on the homotopy class of f. If f has no fixed points, then $\Lambda(f) = 0$. If f has only isolated fixed points (and so a finite number of fixed points), then $\Lambda(f)$ is the number of fixed points counted with multiplicity, which is defined as follows. Let X be a smooth manifold and x a fixed point of f. Consider $1 - f' : X_x \to X_x$. If $\det(1 - f') > 0$, then f has multiplicity $+1$ at x: if $\det(1 - f') < 0$, then the multiplicity is -1. We do not need to discuss the case $\det(1 - f') = 0$.

To compute $\Lambda(f)$ we may replace f with any homotopic map f_0. So we may replace g with any other $g_0 \in G$, since G is path-connected. Take g_0 to be a generator of T (4.1), and



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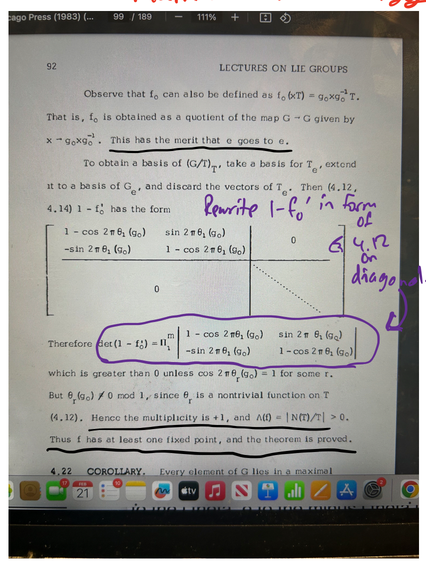
let f_0 be the corresponding map. Then the fixed points of f_0 are the cosets nT for n in $N(T)$, the normaliser of T in G (as the reader will easily verify). Let us examine $N(T)$.

$N(T)$ is a closed subgroup of G, and so is a Lie group (2.27, 2.26), and the identity component $N(T)_1$ is open and so has only a finite number of cosets. Now $N(T)_1 = T$, which we see as follows. $N(T)$ acts on T by conjugation (i.e., $n(t) = ntn^{-1}$) and Aut T is discrete, so $N(T)_1$ acts trivially. (The reader should verify that $N \to \operatorname{Aut} T$ is continuous with this topology on Aut T. Note that this map arises from the map $N \times T \to T$ which is a restriction of the map $G \times G \to G$ given by $(g,h) \to ghg^{-1}$). If $N(T)_1$ properly contains T it contains a 1-parameter subgroup not contained in T but computing with T, contradicting the maximality of T. It follows that $N(T)_1 = T$, that T has only a finite number of cosets in $N(T)$, and that f_0 has only a finite number of fixed points.

It suffices to consider just one of these fixed points, say T, as follows. Let nT be another fixed point. Define $r_n : G/T \to G/T$ by $r_n(gT) = gTn$. This is a well-defined diffeomorphism, commutes with f_0, and takes T to nT. Thus the multiplicity at nT is the same as at T.



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Observe that f_0 can also be defined as $f_0(xT) = g_0 x g_0^{-1} T$. That is, f_0 is obtained as a quotient of the map $G \to G$ given by $x \to g_0 x g_0^{-1}$. This has the merit that e goes to e.

To obtain a basis of $(G/T)_T$, take a basis for T_e, extend it to a basis of G_e, and discard the vectors of T_e. Then (4.12, 4.14) $1 - f_0'$ has the form

Rewrite $1-f_0'$ in form of 4.12 on diagonal.

$$\left[\begin{array}{cc|c} 1 - \cos 2\pi\theta_1(g_0) & \sin 2\pi\theta_1(g_0) & 0 \\ -\sin 2\pi\theta_1(g_0) & 1 - \cos 2\pi\theta_1(g_0) & \\ \hline 0 & & \ddots \end{array}\right]$$

Therefore $\det(1 - f_0') = \prod_1^m \begin{vmatrix} 1 - \cos 2\pi\theta_1(g_0) & \sin 2\pi\theta_1(g_0) \\ -\sin 2\pi\theta_1(g_0) & 1 - \cos 2\pi\theta_1(g_0) \end{vmatrix}$

which is greater than 0 unless $\cos 2\pi\theta_r(g_0) = 1$ for some r. But $\theta_r(g_0) \not\equiv 0 \bmod 1$, since θ_r is a nontrivial function on T (4.12). Hence the multiplicity is $+1$, and $\Lambda(f) = |N(T)/T| > 0$. Thus f has at least one fixed point, and the theorem is proved.

4.22 COROLLARY. Every element of G lies in a maximal

Homotopic Map: A map in which one continuous function gets deformed into another.

Homotopy Class: An equivalence class of continuous functions between two topological spaces which can be deformed into each other.

Prop (2.26) If G is a Lie Group, and H is both a submanifold and a subgroup, then H is a Lie Group.

Prop (2.27) A closed subgroup of H of a Lie Group G is a submanifold.

$\Lambda(f)$ - number of fixed points colloquially.

- This theorem was very grueling but it is good to have proven since we get many corollaries from it.

Corollary (4.22) Every element of G lies in a maximal torus, since the conjugate of a maximal torus is a maximal torus.

Corollary (4.23) Any two maximal tori, T, U are conjugate.

Pf Let u be a generator of U. Then, $u \in xTx^{-1}$ for some $x \in G$, and thus $U \subset xTx^{-1}$. But U is a maximal torus, so $U = xTx^{-1}$. $\square$

Takeaway:

Hence, every construction apparently dependent on a choice of T is independent of the choice up to an inner automorphism of G.

- It is clear that any two maximal tori have the same dimension, which is something that I have been alluding to.

Def (4.24) The dimension of the maximal tori is called the rank of G.

Prop (4.25) Let S be a connected Abelian subgroup of G, and let $g \in G$ commute with all elements of S. Then there is a torus containing both g and S.

(Proof omitted due to confusion).

Not too problematic since there is a more important result.

• Aids you in proving the following:

<u>Prop (4.26)</u> Let T be a maximal torus of G.
If $T \subset A \subset G$ where A is abelian, then $T = A$.
In other words, a maximal torus is a maximal abelian subgroup.

<u>Pf</u> Let $g \in A$. Then by the previous proposition, there is a torus, call it U, that contains both g and T. But T is maximal. Hence, $U = T$ and $g \in T$. Therefore, $A \subset T$. □

<u>Ex</u>: If $a \in U(n)$ commutes with all diagonal matrices, then a itself is diagonal.

<u>Rmk</u>: That being said, it is not, in general, true that a maximal abelian subgroup is in fact a torus.

<u>Ex</u>: Let $G = SO(n)$. $\left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\}$ form a maximal abelian subgroup.

<u>Def (4.29)</u> Let T be a maximal torus of G.
Then, the <u>Weyl Group</u> W (or Φ) of G is the group of automorphisms of T which are restrictions of inner automorphisms of G. This is independent of the choice of T.

• Any such automorphism has the form $t \to ntn^{-1}$, $n \in N(T)$.
$N(T)$ is a closed subgroup of G, and so compact.

• Let $Z(T)$ be <u>the centralizer of T</u>, that is, the set of $z \in G$ such that $ztz^{-1} = t$ $\forall t \in T$. $Z(T)$ is also closed, and $T \subset Z(T) \subset N(T)$. Thus, $N(T)$ maps onto $N(T)/Z(T) \cong W$. $N(T)/T$ is finite, so W is finite (See 4.21).

• Since we are considering G connected,
$Z(T) = T$ and $W = N(T)/T$ (See 4.25).